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B.Sc-part-I(H). paper-I Date-28-04-2020

Algebra:

Q.1. Define homomorphism of a Group. State and prove fundamental theorem of homomorphism groups.

Soln: Homomorphism:→ Let $(G, o)$ and $(G', o')$ be two groups. A mapping $f: G \to G'$ satisfying $f(aob) = f(a) o' f(b), \forall a, b \in G$ is called a homomorphism of $G$ to $G'$.

Statement: Every homomorphic image of a Group $G$ is isomorphic to some quotient Group of $G'$.

Proof: Let $G'$ be a homomorphic image of $G$ and $f$ be the homomorphism of $G$ onto $G'$.

Let $K = \text{Ker} f$

If $a \in G$, then $Ka \in G/K$ and $f(a) \in G'$.

When we definine a mapping $\phi: G/K \to G'$ such that $\phi(Ka) = f(a)$ $\forall a \in G$

$\phi$ is well-defined. Now, we have to prove that $\phi$ is well defined. i.e. if $a, b \in G$ and $Ka = Kb$, then, $\phi(Ka) = \phi(Kb)$

(I) $Ka = Kb \Rightarrow ab^{-1} \in K(a) ?, Kb^{-1}$

$\Rightarrow f(ab^{-1}) = e' \Rightarrow f(a) f(b)^{-1} = e'$

$\Rightarrow f(a)[f(b)]^{-1} = e' \Rightarrow f(a)[f(b)]^{-1} \Rightarrow f(b) = e' f(b)$

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Rest part of question (1)

$\Rightarrow f(a)e' = f(b) \Rightarrow f(a) = f(b)$

$\Rightarrow \phi(Ka) = \phi(Kb)$

$\Rightarrow \phi$ is well-defined

When $\phi$ is one-one. Now, we have

$$\phi(Ka) = \phi(Kb) \Rightarrow f(a) = f(b)$$

$\Rightarrow f(a)[f(b)]^{-1} = f(b)[f(b)]^{-1}$

$\Rightarrow f(a).f(b^{-1}) = e' \Rightarrow f(ab^{-1}) = e'$

$\Rightarrow ab^{-1} \in K$ [$\because K$ is Kernel]

$\Rightarrow Ka = Kb$

$\therefore \phi$ is one-one

When, $\phi$ is onto.

Let $y \in G'$, then $y = f(a)$, for some $a \in G$ as $f$ is onto $G'$.

Now, $Ka \in G/K$ and we have $\phi(Ka) = f(a) = y$

$\Rightarrow \phi$ is onto.

Therefore, $\phi$ is an isomorphism of $G/K$ onto $G'$.

Hence, $G/K \cong G'$ Proved.

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Differential Calculus (Tangents and Normals)

Q.1 Tangent at any point of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$ meets the Co-ordinate axes in A and B. Show that the locus of the point with (OA, OB) as co-ordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Soln: Parametric equations of the curve are

$$x = a\cos^3\theta,\ y = b\sin^3\theta$$

$$\therefore \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta.\ \frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta$$

Here, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{3b\sin^2\theta\cdot\cos\theta}{3a\cos^2\theta\cdot\sin\theta} = -\frac{b}{a}\frac{\sin\theta}{\cos\theta}$

$\therefore$ eqn of the tangent at $(x, y)$ is

$$Y - y = -\frac{b}{a}\frac{\sin\theta}{\cos\theta}(X - x)$$

$$\Rightarrow Y - b\sin^3\theta = -\frac{b}{a}\frac{\sin\theta}{\cos\theta}(X - a\cos^3\theta)$$

$$\Rightarrow a\cos\theta\cdot Y + b\cdot\sin\theta\cdot X = ab\sin\theta\cdot\cos\theta(\sin^2\theta + \cos^2\theta)$$

$$= ab\sin\theta\cdot\cos\theta$$

$$\Rightarrow \frac{X}{a\cos\theta} + \frac{Y}{b\sin\theta} = 1$$

$\therefore$ OA $= a\cos\theta$ and OB $= b\sin\theta$, So, we are required to find the locus of $(a\cos\theta, b\sin\theta)$.

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Rest part of Q.1.

Let $\alpha = a\cos\theta$, $\beta = \sin\theta$

$\therefore \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1$

$\therefore$ Locus of $(\alpha, \beta)$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Q.2. Show that the portion of the tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ which is intercepted between the axes, is of constant length and find the area of the portion included between the axes and tangent.

Soln: The equation of the tangent at $(x, y)$ is

$$(X - x) f_x + (Y - y) f_y = 0$$

The eqn of the curve is

$$f(x,y) = x^{2/3} + y^{\frac{2}{3}} - a^{2/3} = 0.$$

$\therefore f_x = \frac{2}{3} x^{-\frac{1}{3}}, \; f_y = \frac{2}{3} y^{-\frac{1}{3}}$

$\therefore$ The eqn. of the tangent becomes

$$X \frac{2}{3} x^{-1/3} + Y \frac{2}{3} y^{-\frac{1}{3}} = x \frac{2}{3} x^{-\frac{1}{3}} + y \frac{2}{3} y^{-\frac{1}{3}}$$

$$= \frac{2}{3}\left(x^{\frac{2}{3}} + y^{\frac{2}{3}}\right) = \frac{2}{3} a^{\frac{2}{3}}$$

$\therefore \frac{X}{x^{1/3}} + \frac{Y}{y^{1/3}} = a^{2/3} \Rightarrow \frac{X}{a^{\frac{2}{3}} x^{1/3}} + \frac{Y}{a^{2/3} y^{1/3}} = 1$

$\therefore$ Intercepts $OA = a^{\frac{2}{3}} x^{1/3}$ and $OB = a^{\frac{2}{3}} y^{1/3}$

$\therefore$ Portion of the tangent intercepted between the axes $= AB = \sqrt{OA^2 + OB^2} = \sqrt{a^{\frac{4}{3}} x^{\frac{2}{3}} + a^{\frac{4}{3}} y^{\frac{2}{3}}} = a^{\frac{2}{3}} \sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}} = a^{\frac{2}{3}} . a^{\frac{1}{3}}$
$= a.$

Which is constant.

Area of the portion included between the axes and the tangent = Area of the $\Delta OAB$

$$= \frac{1}{2} \cdot OA \cdot OB = \frac{1}{2} a^{\frac{2}{3}} . x^{\frac{1}{3}} . a^{\frac{2}{3}} y^{\frac{1}{3}} = \frac{1}{2} a^{\frac{4}{3}} x^{1/3} y^{1/3}$$

Ans.

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